Finite-size corrections for the low-lying states of the integrable model with two- and three-particle interactions

LETTER TO THE EDITOR

Finite-size corrections for the low-lying states of the integrable model with two- and three-particle interactions

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Abstract. The finite-size corrections to the ground state and the energy of the low-lying states as a function of the size L are calculated for the one-dimensional integrable model with two- and three-particle interactions. It is shown that the conformal structure can be recovered if some extra conditions are imposed on the size and the ratio of the interaction parameters of the system. In this case the conformal properties of model are the same as those of the isotropic Heisenberg chain.

The calculation of finite-size corrections to the free energy of a lattice spin system and the ground state energy and low lying part of the spectrum of a quantum spin chain has become a subject of considerable interest in the theory of integrable two-dimensional models. It is related with the fact that, due to development of the concept of conformal invariance [1, 2] it was shown [3–5] that the conformal anomaly and the scaling dimensions of primary conformal order parameters are directly accessible through the finite-size corrections of an affiliated system defined on infinitely long but finitely wide strips at criticality. Therefore the calculation of finite-size corrections permits us to determine these important parameters describing the critical behaviour of the system. Such a program was done for most integrable systems of classical and quantum statistical physics [6–15].

In previous papers [16, 17] we proposed a new integrable spin chain with two- and three-particle interactions and calculated exactly the ground state energy and the spectrum of excitations. In the present work we report the results of calculations of the finite-size corrections to these quantities.

The model under consideration is described by the Hamiltonian

$$H = - \sum_{j=1}^{L} \sum_{\tau=1,2} (c_{j(\tau)}^{+} c_{j+1(\tau)} + c_{j+1(\tau)}^{+} c_{j(\tau)})$$

$$+ U \sum_{j=1}^{L} \sum_{\tau=1,2} (c_{j(\tau)}^{+} c_{j+1(\tau)} + c_{j+1(\tau)}^{+} c_{j(\tau)}) n_{j+\tau-1(\tau+1)} \tag{1}$$

where $c_j(\tau)$ are electron creation and destruction operators on the sublattice τ ($\tau = 1, 2, c_{j(3)} \equiv c_{j(1)}$), and $n_{j(\tau)} = c_{j(\tau)}^{+} c_{j(\tau)}$ is the corresponding number operator. This

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Hamiltonian can be diagonalized by solving the set of Bethe ansatz equations [16]:

$$Lk_j + \sum_{\beta=1}^{m} \theta(k_j - \Lambda_\beta; \alpha') = 2\pi I_j \qquad j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} \theta(\Lambda_\beta - k_j; \alpha') - \sum_{\gamma=1}^{m} \theta(\Lambda_\beta - \Lambda_\gamma; 2\alpha') = 2\pi J_\beta \qquad \beta = 1, 2, \ldots, m$$

$$\theta(k; \alpha') = 2 \tan^{-1}(\coth \alpha' \tan \tfrac{1}{2}k) \qquad -\pi \le \theta(k; \alpha) < \pi$$

$$e^\alpha = (1 - U)^{-1} \qquad 2\alpha' = \alpha \qquad 0 < U < 1.$$

(2)

Here k_j are the momenta of the electrons and Λ_β are connected with the particle distribution between sublattices. I_j and J_β are integer (half-integer) numbers for even (odd) m and $n - m - 1$ respectively.

A solution of (2) corresponds to an eigenstate which is characterized by the total number of electrons n and the number of electrons on the first sublattice m. The energy and the momentum of this state are, respectively

$$E = -2 \sum_{j=1}^{n} \cos k_j$$

(3)

$$P = \sum_{j=1}^{n} k_j = \frac{2\pi}{L} \left(\sum_{j=1}^{n} I_j + \sum_{\beta=1}^{m} J_\beta \right).$$

(4)

It was shown in a previous paper [17] that the excitations corresponding to the redistribution of particles between sublattices have a gap. This means that its contribution in the finite-size corrections are exponentially small. Therefore we choose the numbers J_β as in the ground state

$$J_\beta = \beta - (m+1)/2 \qquad \beta = 1, 2, \ldots, m \qquad m = n/2.$$

(5)

This corresponds to the symmetrical case where we have the same number of particles on each sublattice. Moreover, we may replace in (2) Σ_β by $L \int_{-\pi}^{\pi} d\Lambda\, \sigma(\Lambda)$ where $\alpha(\Lambda)$ is the density of values Λ_β. Since the period of integration on Λ is a complete period we get the following equations by Fourier transforming (2)

$$Lk_j + \sum_{j=1}^{n} \theta(k - k_j) = 2\pi I_j$$

(6)

where $\theta'(k) = \mathcal{S}(k)$ is the function which is defined in [16]

$$\mathcal{S}(k) = \tfrac{1}{2} + 2 \sum_{n=1}^{\infty} \cos(nk)/[1 + \exp(2|\alpha|n)].$$

(7)

We choose two numbers I^+ and I^- both equal to $n/4 \bmod 1$, so that $I^+ - I^- = n$ and $\frac{1}{2}(I^+ - I^-) = n$ and $\frac{1}{2}(I^+ + I^-) = d$. For I_j we take all the numbers equal to $(n/2) + 1)/2 \bmod 1$ between I^+ and I^-. It corresponds to a Fermi point to the right one.

The equations (6) have the same structure of the corresponding equations in the Heisenberg chain. Therefore in calculating the finite-size effects we closely follow the method developed in [14, 15]. We define the functions

$$X(k) = \frac{1}{2\pi} \left[k + \frac{1}{L} \sum_{j=1}^{n} \theta(k - k_j) \right] \qquad \rho_L(k) = \frac{dX(k)}{dk}.$$

(8)

With these definitions the Bethe ansatz equations (2) take the form

$$X(k_j) = \frac{1}{L} I_j.$$ (9)

Using the Euler–Maclaurin formula

$$\frac{1}{L} \sum_j f(k_j) = \int_{k^-}^{k^+} f(k)\rho_L(k)\,dk + \frac{1}{24L^2}\left\{\frac{f'(k^-)}{\rho_L(k^-)} - \frac{f'(k^+)}{\rho_L(k^+)}\right\} + \mathcal{O}\left(\frac{1}{L^3}\right)$$ (10)

equations (8) and (9) can be written in the form:

$$\rho_L(k) = \frac{1}{24}\left\{1 + 1\int_{k^-}^{k^+} \varphi(k - k')\rho_L(k')\,dk' + \frac{1}{24L^2}\left(\frac{\varphi'(k - k^+)}{\rho_L(k^+)} - \frac{\varphi'(k - k^-)}{\rho_L(k^-)}\right)\right.$$ (11)

where the integration boundaries $k^\pm$ are determined by

$$X(k^\pm) = I^\pm/L.$$ (12)

The linear integral equation (11) is completed by the equations determining k^+ and k^- which are obtained from (8) and (12)

$$\int_{k^-}^{k^+} \rho_L(k)\,dk = n/L$$

$$\frac{1}{2}\left\{-\int_{k^-}^{0} \rho_L(k)\,dk + \int_{0}^{k^+} \rho_L(k)\,dk\right\} = d,$$ (13)

Due to the linearity of (11) ρ_L can be written in the following form

$$\rho_L(k) = \rho(k\,|\,k^+, k^-) + \frac{1}{24L^2}\left\{\frac{q(k\,|\,k^+, k^-)}{\rho_L(k^+)} \times \frac{q(-k\,|\,-k^-, -k^+)}{\rho_L(k^-)}\right\}$$ (14)

where $\rho(k\,|\,k^+, k^-)$ and $q(k\,|\,k^+, k^-)$ are the solutions of the following integral equations

$$\rho(k\,|\,k^+, k^-) = \tfrac{1}{2}\pi\left\{1 + \int_{k^-}^{k^+} \varphi(k - k')\rho(k'\,|\,k^+, k^-)\,dk'\right\}$$ (15)

$$q(k\,|\,k^+, k^-) = \tfrac{1}{2}\pi\left\{\varphi'(k - k^+) + \int_{k^-}^{k^+} \varphi(k - k')q(k'\,|\,k^+, k^-)\,dk'\right\}$$ (16)

Equations (13)–(16) form a closed system and determine completely the state under consideration. The energy of this state according to (3), (10) and (14) is

$$\frac{1}{L} E = \varepsilon(k^+, k^-) - \frac{1}{12L^2}\left\{\frac{e(k^+, k^-)}{\rho_L(k^+)} + \frac{e(-k^+, -k^-)}{\rho_L(k^-)}\right\}$$ (17)

with

$$\varepsilon(k^+, k^-) = \int_{k^-}^{k^+} \varepsilon_0(k)\rho(k\,|\,k^+, k^-)\,dk \qquad \varepsilon_0(k) = -2\cos k$$ (18)

and

$$e(k^+, k^-) = \sin k^+ + \int_{k^-}^{k^+} \cos k\, q(k\,|\,k^+, k^-)\,dk.$$ (19)

The system is in the ground state if the energy in (17) is minimal with respect to n/L and d/L. The procedure of minimization can be performed by the method used previously for the Heisenberg chain [14] and for the one-dimensional Hubbard model [15]. It leads to the following result. At the minimum we have $k^+ = -k^- = Q$, $\varepsilon(k^+, k^-) = \varepsilon_\infty$ (the energy density of the infinite system). Near this point the energy of the finite system (17) has the following form

$$\frac{1}{L} E^{(0)} = \varepsilon_\infty + \frac{1}{L^2} \frac{2e(Q, -Q)}{\rho_\infty(Q)} \left\{ \frac{(n - \eta L)^2}{[4\pi\rho_\infty(Q)]^2} + \left[\frac{\rho_\infty(Q)}{\rho_\infty(0)} \right]^2 d^2 - \frac{1}{12} \right\} + \mathcal{O}\left(\frac{1}{L^3}\right). \tag{20}$$

The momentum of the considered state is

$$\rho^{(0)} = \frac{2\pi}{L} nd. \tag{21}$$

In these expressions $\rho_\infty(k)$ is the density of roots for the infinite system which was determined in [16] as a solution of integral equation, i.e. equation (14) for $k^+ = -k^- = Q$ and

$$\eta = \int_{-Q}^{Q} \rho_\infty(k) \, dk. \tag{22}$$

Equation (20) gives the energy of the ground state if $n - mL$ is minimal. Analogous calculations can be performed for the finite-size corrections to the energy of the low-lying states where excitations are introduced in the vicinity of the Fermi points. We characterize the holes and particles in the vicinity of I_n^+ and I_p^-. For the energy and momentum of this state we obtain

$$\frac{1}{L} E = \frac{1}{L} E^{(0)} + \frac{1}{L^2} \frac{2e(Q, -Q)}{\rho_\infty(Q)} (N^+ + N^-) \tag{23}$$

$$p = p^{(0)} + \frac{2\pi}{L} (N^+ - N^-) \tag{24}$$

where

$$N^+ = \sum_p I_p^+ - \sum I_h^+ \qquad N^- = \sum_h I_h^- - \sum I_p^+. \tag{25}$$

The expressions for the ground state (20) and for the spectrum of low lying excitations are not of the conformal form since the coefficient of the term proportional to $1/L^2$ is still dependent on L for the arbitrary value η. This phenomenon is known already for the Heisenberg chain in the external field [14] and is thought to be connected with the possibility of a consistent definition of the continuum limit for the system. In our case the spectrum will be analytic if η is rational. Then we can choose $n = \eta L$ in the expression for the energy (20) and $n = \Delta n + \eta L$ in the expression for the excited states (23). In this case the expressions (20) and (23) have the conformal anomaly c and the conformal dimensions (X, S) of the operators associated with these excited states: one has simply to compare (20) and (23) with the predictions of the conformal theory [3-5, 13]. We have $c = 1$ and

$$X(\Delta n, d) = \left[\frac{\Delta n}{4\pi\rho_\infty(Q)} \right]^2 + \left[\frac{\rho_\infty(Q)}{\rho_0(Q)} \right]^2 d^2$$

$$s(\Delta n, d) = \Delta nd. \tag{26}$$

These formulae are analogous to those for the Heisenberg chain in the external field [14]. There, however, such abnormal behaviour of the Heisenberg chain is the

result of the presence of a magnetic field. In our case it is the internal property of the model we consider. It can be expected that a generalized soluble model containing an arbitrary number of sublattices [18] has an analogous behaviour.

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